SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 1
MARCH 2014

Mathematics Extension 2

Name ....

Teacher ....

General Instructions

- Working Time - 70 min.
- Write using a blue or black pen.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 6-9, show relevant mathematical reasoning and /or calculations.
- Begin each question on a fresh sheet of paper.

Total marks (53)

- Attempt Questions 1-9.
- All questions are of equal value.

Section 1
Multiple Choice (5 marks)
Use the multiple choice answer sheet for Question 1-5

1. In the Argand Diagram the locus of the point P representing the complex number z such that $|z - 1 + i| = 4$ is a circle. What are the centre and radius of this circle?

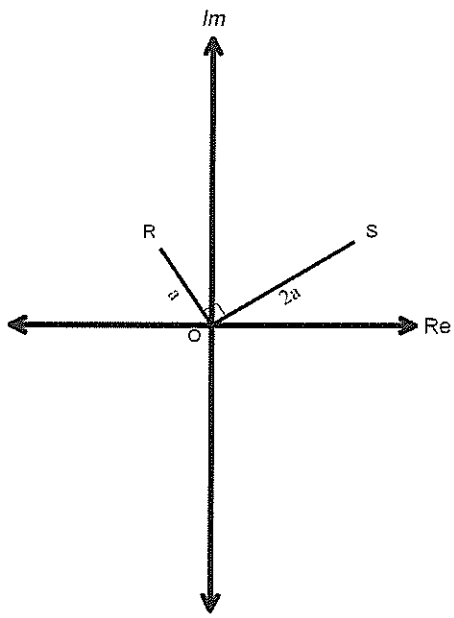
   (A) centre (-1,1), radius 4

   (B) centre (-1,1), radius 2

   (C) centre (1,-1), radius 4

   (D) centre (1,-1), radius 2

2. In the Argand Diagram below the points R and S represent the complex numbers w and z respectively, where $\angle ROQ = 90°$. The distance OS is 2a units and the distance OR is a units. Which of the following is correct?



   (A) $w = 2iz$

   (B) $\mathbf{w} = \mathbf{i}\overline{w}$

   (C) $w = -\dfrac{iz}{2}$

   (D) $w = -\dfrac{z}{2i}$

3. Let $z = a + ib$ where $a \neq 0$ and $b \neq 0$.
Which of the following statements is false.

(A) $z - \overline{z} = 2bi$

(B) $|z|^2 = |z||\overline{z}|$

(C) $|z| + |\overline{z}| = |z + \overline{z}|$

(D) $\arg(z) + \arg(\overline{z}) = 0$

4. Which pair of equations gives the directrices of $4x^2 - 25y^2 = 100$

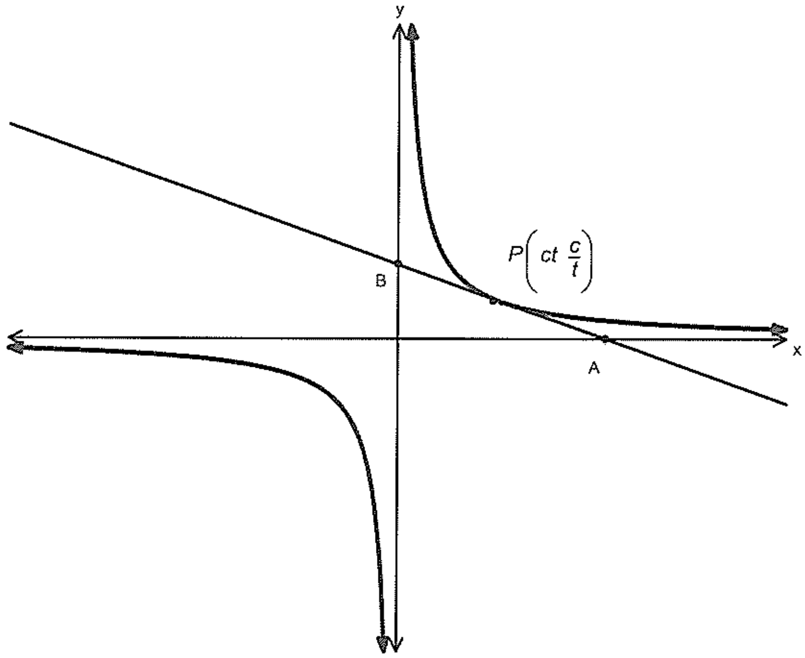
(A) $x = \pm\frac{25}{\sqrt{29}}$

(B) $x = \pm\frac{1}{\sqrt{29}}$

(C) $x = \pm\sqrt{29}$

(D) $x = \pm\frac{\sqrt{29}}{25}$

5. The equation of the tangent to the rectangular hyperbola $xy = c^2$ at $P\left(ct, \frac{c}{t}\right)$ is given by $x + t^2 y = 2ct$. The tangent cuts the x and y axes at A and B respectively.



Which of the following statements is false?

(A) P is the centre of the circle that passes through O, A and B.

(B) The area of ΔAOB is $2c^2$ square units

(C) The distance AB is $\sqrt{4c^2 t^2 + \frac{4c^2}{t^2}}$.

(D) $AP > BP$

Section II

Total Marks (48)
Attempt Questions 6 – 9.

Answer each question in your writing booklet.
In Questions 6-9, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 **(9 Marks)** Use a Separate Sheet of paper

a) Given $A = 3 - 4i$ and $B = 5 + 3i$, express the following in the form $x + iy$ where x and y are real numbers.

i. $\frac{A}{B}$ **2**

iiv. $\sqrt{A}$ **2**

b) Show that the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $P(x_0 , y_0)$ has equation: $\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$. **2**

c) On an Argand diagram, sketch the region where the inequalities

$2 \le |z| \le 5$ and $\arg \frac{\pi}{6} < \arg z < \frac{2\pi}{3}$ hold simultaneously **3**

End of Question 6

Question 7 **(9 Marks)** Use a Separate Sheet of paper

a) i. Find the five fifth roots of $z^5 = 1$ and plot these on the Argand diagram. **2**

ii. Prove that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$. **3**

b) i. Expand $(\cos\theta + i\sin\theta)^3$ using Pascals triangle (or other) **1**

ii. Expand $(\cos\theta + i\sin\theta)^3$ using de Moivres theorem and hence show that

$$\sin 3\theta = 3\sin\theta - 4sin^3\theta$$ **3**

End of Question 7

Question 8 **(9 Marks)** Use a Separate Sheet of paper

a) A conic C has foci at (4,0) and (-4, 0) and has eccentricity, $e = \sqrt{2}$.
Find the equation of this conic **2**

b) i) Show that $P(2\sqrt{2}\cos\theta,\ 3\sqrt{2}\sin\theta)$ lies on the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 2$ **1**

ii) Show that the slope of the tangent at P is $\frac{-3\cos\theta}{2\sin\theta}$ **2**

iii) Find the equation of the normal to the ellipse at P **2**

iv) Find the value of θ to the nearest degree, if the normal passes through the point $(-2\sqrt{2}, 0)$ **2**

End of Question 8

Question 9 **(9 Marks)** Use a Separate Sheet of paper

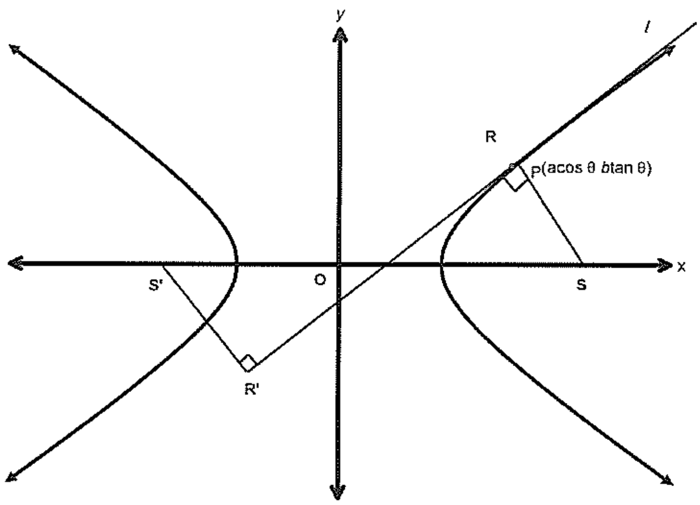
a) $z_1 = 1 + i\sqrt{3}$ and $z_2 = 1 - i$ are two complex numbers.

i. Express z_1, z_2 and $\frac{z_1}{z_2}$ in modulus/argument form. **2**

ii. Find the smallest positive integer n such that $\frac{(z_1)^n}{(z_2)^n}$ is imaginary.

For this value of n, write the value of $\frac{(z_1)^n}{(z_2)^n}$ in the form bi where b is a real number. **2**

b) Let $P(a\cos\theta, b\tan\theta)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ where $a > 0$ and $b > 0$ as shown in the diagram. The foci of the hyperbola are S and S', l is the tangent to the point P.



The points R and R' lie on l so that SR and $S'R'$ are perpendicular to l.

The line l has equation $bx\sec\theta - ay\tan\theta - ab = 0$ **2**

i. Show that $$SR = \frac{ab(e\sec\theta - 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$$ **2**

ii. Show that $SR \times S'R' = b^2$ **3**

End of Examination

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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Mathematics Extension 2

Assessment Task 1

March 2014

Multiple Choice

1. C
2. D
3. C
4. A
5. D

Question 6

a i) $\frac{A}{B} = \frac{3-4i}{5+3i}$

$= \frac{3-4i}{5+3i} \times \frac{5-3i}{5-3i}$

$= \frac{15-9i-20i-12}{25+9}$

$= \frac{3-29i}{34}$

$= \frac{3}{34} - \frac{29}{34}i$

a ii) let $\sqrt{A} = x + iy$

$\therefore A = x^2 - y^2 + 2xyi$

$x^2 - y^2 = 3 \quad \ldots\ldots (1)$

$2xy = -4$

$(x^2+y^2)^2 = (x^2-y^2)^2 + 4x^2y^2$

$= 3^2 + 4^2$

$= 25$

$x + y^2 = 5 \quad \ldots\ldots (2)$

$(1) + (2) = 2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$(2) - (1) = 2y^2 = 2$

$y^2 = 1$

$y = \pm 1$

Since $2xy = -4$

$\sqrt{A} = \pm(2 - i)$

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b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{2y}{b^2}\frac{dy}{dx} = \frac{-2x}{a^2}$

$\frac{dy}{dx} = \frac{-b^2x}{a^2y}$

At (x_0, y_0) $\quad y - y_1 = m(x - x_1)$

$m = \frac{-b^2x}{a^2y_0}$ $\quad y - y_0 = \frac{-b^2x_0}{a^2y_0}(x - x_0)$

$a^2yy_0 - a^2y_0^2 = -b^2xx_0 + b^2x_0^2$

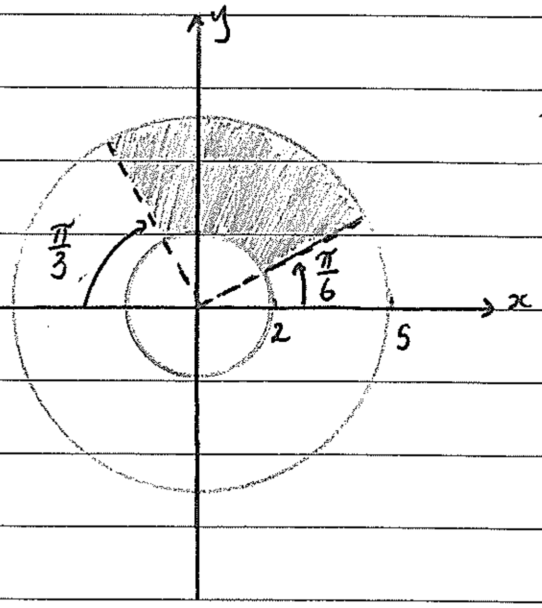
$b^2xx_0 + a^2yy_0 = a^2y_0^2 + b^2x_0^2$

Dividing by a^2b^2

$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2}$

$\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$

$\therefore \frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1$



Student Name: ____________________ Teacher Name: ____________________

Question 7

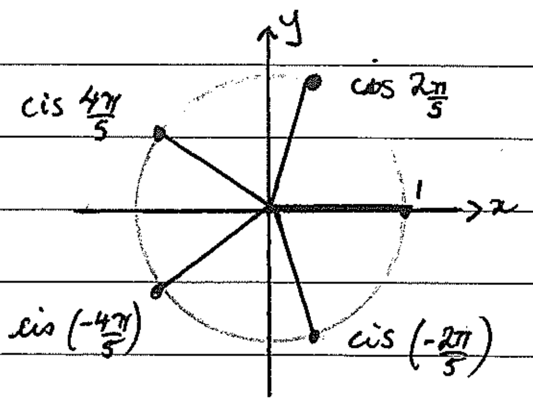
a i) $z_1 = 1$

$z_2 = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$

$z_3 = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}$

$z_4 = \cos\left(-\frac{4\pi}{5}\right) + i\sin\left(-\frac{4\pi}{5}\right)$

$z_5 = \cos\left(-\frac{2\pi}{5}\right) + i\sin\left(-\frac{2\pi}{5}\right)$



a ii) Sum of root of $z^5 - 1 = 0$ is 0

$z_1 + z_2 + z_3 + z_4 + z_5 = 0$

$1 + \text{cis}\frac{2\pi}{5} + \text{cis}\frac{4\pi}{5} + \text{cis}\left(-\frac{4\pi}{5}\right) + \text{cis}\left(-\frac{2\pi}{5}\right) = 0$

since $\text{cis}\frac{2\pi}{5} + \text{cis}\left(-\frac{2\pi}{5}\right) = 2\cos\left(\frac{2\pi}{5}\right)$

$1 + 2\cos\frac{2\pi}{5} + 2\cos\frac{4\pi}{5} = 0$

$2\left(\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5}\right) = -1$

$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$

b i) $(\cos\theta + i\sin\theta)^3$

$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$

b ii) $(\cos\theta + i\sin\theta)^3$

$\cos 3\theta + i\sin 3\theta$

Equating the imaginary parts

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$

$= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta$

$= 3\sin\theta - 3\sin^3\theta - \sin^3\theta$

$= 3\sin\theta - 4\sin^3\theta$

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Question 8

a) Foci $= \pm ae$

$4 = a\sqrt{2}$

$a = \frac{4}{\sqrt{2}} = 2\sqrt{2}$

$b = 2\sqrt{2}$

Rectangular hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{x^2}{8} - \frac{y^2}{8} = 1$

$x^2 - y^2 = 8$

b i) $\frac{x^2}{4} + \frac{y^2}{9} = 2$

$P(2\sqrt{2}\cos\theta, 3\sqrt{2}\sin\theta)$

$L.H.S = \frac{(2\sqrt{2}\cos\theta)^2}{4} + \frac{(3\sqrt{2}\sin\theta)^2}{9}$

$\frac{8\cos^2\theta}{4} + \frac{18\sin^2\theta}{9}$

$2(\cos^2\theta + \sin^2\theta)$

$= 2$

$= R.H.S$

b ii) $\frac{x^2}{4} + \frac{y^2}{9} = 2$

$\frac{x}{2} + \frac{2y}{9}\frac{dy}{dx} = 0$

$\frac{2\sqrt{2}\cos\theta}{2} + \frac{6\sqrt{2}\sin\theta}{9}\frac{dy}{dx} = 0$

$\frac{2\sin\theta}{3}\frac{dy}{dx} = -\cos\theta$

$\frac{dy}{dx} = -\frac{3\cos\theta}{2\sin\theta}$

b iii) $m_2 = \frac{+2\sin\theta}{3\cos\theta}$

$y - 3\sqrt{2}\sin\theta = \frac{2\sin\theta}{3\cos\theta}(x - 2\sqrt{2}\cos\theta)$

$3y\cos\theta - 9\sqrt{2}\sin\theta\cos\theta = 2x\sin\theta - 4\sqrt{2}\sin\theta\cos\theta$

$3y\cos\theta - 2x\sin\theta = 5\sqrt{2}\sin\theta\cos\theta$

b iv) $P(-2\sqrt{2}, 0)$

$3y\cos\theta - 2x\sin\theta = 5\sqrt{2}\sin\theta\cos\theta$

$3\times 0\cos\theta - 2(-2\sqrt{2})\sin\theta = 5\sqrt{2}\sin\theta\cos\theta$

$0 + 4\sqrt{2}\sin\theta = 5\sqrt{2}\sin\theta\cos\theta$

3 solutions

$\sin\theta = \underline{\underline{0^\circ}}$ $\quad 4\sqrt{2}\sin\theta = 5\sqrt{2}\sin\theta\cos\theta$

$\cos\theta = \frac{4}{5}$

$\theta = \underline{37^\circ}, \underline{323^\circ}$

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Question 9

ai $z_1 = 1 + i\sqrt{3}$

$z_2 = 1 - i$

$z_1 = 2\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$\frac{z_1}{z_2} = \frac{2}{\sqrt{2}}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$

aii) $\arg\left(\frac{z_1^{\,n}}{z_2^{\,n}}\right) = \frac{7n\pi}{12}$

$\therefore \left(\frac{2}{\sqrt{2}}\right)^6\left[\cos\left(\frac{7\times 6\pi}{12}\right) + i\sin\left(\frac{7\times 6\pi}{12}\right)\right]$

$\therefore \frac{z_1^{\,n}}{z_2^{\,n}}$ is imaginary if $\frac{7n\pi}{12} = \frac{m\pi}{2}$

The smallest positive integer n is 6

$\arg\left(\frac{z_1^{\,6}}{z_2^{\,6}}\right) = \frac{7\pi}{2} = 4\pi - \frac{\pi}{2} = -\frac{\pi}{2}$

$\left|\frac{z_1^{\,6}}{z_2^{\,6}}\right| = (\sqrt{2})^6 = 8$

$\frac{z_1^{\,6}}{z_2^{\,6}} = 8\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$

$= -8i$

b i) Perpendicular distance

$SR = \frac{|b\times ae\sec\theta - 0 - ab|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$

$= \frac{ab|e\sec\theta - 1|}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$

as $a>0$ $b>0$

$= \frac{ab(e\sec\theta - 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$

as $e>0$ $\sec\theta > 1$

b ii) $S'R' = \frac{|b(-ae)\sec\theta - 0 - ab|}{\sqrt{b^2\sec^2\theta + a^2\tan^2\theta}}$

$= \frac{|-ab(e\sec\theta + 1)|}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$

as $a>0$ $b>0$

$= \frac{ab(e\sec\theta + 1)}{\sqrt{a^2\tan^2\theta + b^2\sec^2\theta}}$

$\therefore SR \times S'R'$

$= \frac{a^2b^2(e\sec\theta - 1)(e\sec\theta + 1)}{a^2\tan^2\theta + b^2\sec^2\theta}$

Student Name: ______________________ Teacher Name: ______________________

$$= \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2\tan^2\theta + b^2\sec^2\theta}$$

$$= \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2\tan^2\theta + a^2(e^2-1)\sec^2\theta}$$

$$b^2 = a^2(e^2-1)$$

$$= \frac{a^2b^2(e^2\sec^2\theta - 1)}{a^2[e^2\sec^2\theta - (\sec^2\theta - \tan^2\theta)]}$$

$$= \frac{b^2(e^2\sec^2\theta - 1)}{(e^2\sec^2\theta - 1)}$$

$$= b^2$$

$$\therefore SR \times S'R' = b^2$$